

University of Groningen

## Trade-offs between Control and Mode-Observability Properties for Switching Linear Systems

Baglietto, Marco; Battistelli, Giorgio; Tesi, Pietro

*Published in:*

Proceedings of the 51st Annual Conference on Decision and Control (CDC)

**IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.**

*Document Version*

Publisher's PDF, also known as Version of record

*Publication date:*

2012

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Baglietto, M., Battistelli, G., & Tesi, P. (2012). Trade-offs between Control and Mode-Observability Properties for Switching Linear Systems. In *Proceedings of the 51st Annual Conference on Decision and Control (CDC)* (pp. 719-724). IEEE (The Institute of Electrical and Electronics Engineers).

### Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

### Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*

# Trade-offs between Control and Mode-Observability Properties for Switching Linear Systems

Marco Baglietto, Giorgio Battistelli and Pietro Tesi

**Abstract**—This paper describes recent progress in the study of switching linear systems *i.e.* linear systems whose dynamics can switch among a family of possible configurations/modes. We focus the attention on *closed-loop mode-observability*, namely the problem of identifying the active (unknown) mode of the system from closed-loop data. The analysis focuses on two fundamental questions: i) How the *control objectives* influence mode-observability; ii) How the *control design* influences mode-observability.

## I. INTRODUCTION

In recent years, the study of switching systems has received a great deal of attention both in theory and applications, as they allow one to describe the behaviour of a large class of plants resulting from the interactions of continuous dynamics, discrete dynamics, and logic decisions [1]. Switching systems represent a special class of hybrid systems, namely those systems whose dynamics can switch among a family of possible configurations/modes. From a theoretical viewpoint, the main contributions to the study of switching systems have been basically of a two-fold nature: on one side several studies have focused on state/mode observability, *viz.* on the possibility of reconstructing from measured data the continuous state, the discrete mode, or both [2]–[6]; on the other side, the main interest has been devoted to stability and stabilization problems [7]–[9]. Within this latter source of contribution, however, the major emphasis has been on basic issues, namely the characterization of the control laws which can ensure stability to the switching system under the assumption that an exact knowledge of the active process mode is available in real-time or with delay.

In many control applications involving switching, however, the plant switching signal is neither known nor observed. Typical examples are found in connection with reconfigurable control systems where multiple models are used to describe plants around different operating points or faulty mode dynamics for systems subjects to failures; see [1], [10]. The departure from the assumption that an exact knowledge of the process mode sequence is available poses major challenges. This is because, in order to properly configure the control action, specific mechanisms have to be devised apt to estimate the current process mode on the grounds of the available data. In this respect, while several

results have been reported for autonomous systems, little is known on how to deal with non-autonomous systems. In particular, in relation with the latter case, most of the relevant literature is concerned with the question of whether the active plant mode can be identified through suitable control input signals [3]–[5]. Such an analysis, however, appears to be mainly motivated by the goal of establishing connections between input selection and mode-identifiability, in close analogy with the problem of input selection for systems parameter identification [11]. In many case, indeed, the question of mode-identifiability (or mode-observability) has to be approached in a different way, of primary practical importance being the issues of stability and performance. The objective of this paper is to investigate the question of mode-observability for switching linear systems in feedback form, by focusing on the following question:

- i) How the *control objectives* influence the possibility of identifying the active mode of the plant.
- ii) How the *control design* influences the possibility of identifying the active mode of the plant.

For switching linear systems in feedback form, trade-offs between mode-observability and control objectives may arise. In particular, it is shown that the presence of control objectives such as asymptotic tracking can make the plant mode identification task impossible to solve, thus revealing the possible existence of conflicting objectives. It is also shown, nonetheless, that suitable control design conditions do exist under which it is possible to ensure mode-observability along all closed-loop trajectories except those corresponding to steady-state tracking. For preliminary results on the subject see [12], [13].

**Notations.** Given a matrix  $M$ ,  $M^\top$  is its transpose and  $\|M\| = [\lambda_{\max}(M^\top M)]^{1/2}$  its norm, where  $\lambda_{\max}$  denotes the maximum eigenvalue. Given a measurable time function  $v : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and a time interval  $\mathcal{I} \subseteq \mathbb{R}_+$ , we denote its  $\mathcal{L}_2$  and  $\mathcal{L}_\infty$  norm on  $\mathcal{I}$  as  $\|v\|_{2,\mathcal{I}} = \sqrt{\int_{\mathcal{I}} |v(t)|^2 dt}$  and  $\|v\|_{\infty,\mathcal{I}} = \text{ess sup}_{t \in \mathcal{I}} |v(t)|$  respectively.

## II. PROBLEM OVERVIEW

Consider a family of dynamical systems

$$\begin{cases} \dot{x} = A_i x + B_i u \\ y = C_i x \end{cases} \quad i \in \mathcal{N}, \quad (1)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the input,  $y \in \mathbb{R}^{n_y}$  is the output and  $\mathcal{N} := \{1, 2, \dots, N\}$  is a finite index set.  $A_i$ ,  $B_i$ , and  $C_i$ ,  $i \in \mathcal{N}$ , are constant matrices of appropriate

M. Baglietto and P. Tesi are with the Department of Communications, Computer and System Sciences, University of Genoa, Via Opera Pia 13, 16145 Genova, Italy mbaglietto@dist.unige.it, pietro.tesi@unige.it

G. Battistelli is with the Dipartimento Sistemi e Informatica, Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy battistelli@dsi.unifi.it

dimensions. We denote by *switching plant* the dynamical system generated by the family (1) along with a switching signal  $\rho : \mathbb{R}_+ \rightarrow \mathcal{N}$ , belonging to the class of functions that are piecewise constant, right continuous, and specifying at every time the index of the active system.

For systems subject to large modeling uncertainty, there may be no single controller that achieves satisfactory performance for all possible plants configurations. In this case, it is common practice to design a family of candidate controllers so that each system in (1) performs satisfactorily when controlled by at least one of the candidate controllers. Then, a suitably designed high-level unit orchestrates the switching among the controllers so as to ensure desired closed-loop stability and performance properties.

In order to make this precise, let

$$\begin{cases} \dot{q} = F_j q + G_j e \\ u = H_j q + K_j e \end{cases} \quad i \in \mathcal{M}, \quad (2)$$

be a family of candidate controllers, where  $q \in \mathbb{R}^{n_q}$  is the state and  $e = r - y$  is the tracking error, where  $r$  is a reference signal to be tracked by the plant output;  $\mathcal{M} := \{1, 2, \dots, M\}$  is a finite index set;  $F_j, G_j, H_j$ , and  $K_j, j \in \mathcal{M}$ , are constant matrices of appropriate dimensions. We denote by *switching controller* the dynamical system generated by the family (2) along with a switching signal  $\sigma : \mathbb{R}_+ \rightarrow \mathcal{M}$  continuous from the right and specifying at every time the index of the active controller. Throughout the paper, we shall assume that  $r$  is generated by a finite-dimensional linear time-invariant system  $\mathcal{E}$  (*exosystem*),

$$\begin{cases} \dot{p} = E p \\ r = L p \end{cases}, \quad (3)$$

where  $p \in \mathbb{R}^{n_p}$  is the state and  $r \in \mathbb{R}^{n_u}$ .

The closed-loop composed by the switching plant, the switching controller and the exosystem is depicted in Fig. 1. Hereafter,  $\mathcal{P}_i$  and  $\mathcal{C}_j$  will denote the LTI systems associated with the  $i$ -th plant and the  $j$ -th controller, respectively.

#### A. Closed-loop Issues

In many control applications involving switching, the plant switching signal is *unknown* and *unobserved*, thus meaning that an exact knowledge of the current plant mode is available neither in real-time nor with delay. Accordingly, in order to properly configure the control action, specific mechanisms have to be devised apt to estimate the current plant mode on the grounds of the available data. Due to its practical importance, in recent years, the problem of identifying the active mode of the plant from observations has therefore attracted significant research efforts (e.g. see [2], [3], [6]). In this respect, while several results have been reported for autonomous systems, little is known for configurations such as the one depicted in Fig. 1. In fact, most of the analysis for non-autonomous systems is concerned with the question of whether the active plant mode can be identified through suitable control input signals. Such an analysis, however,

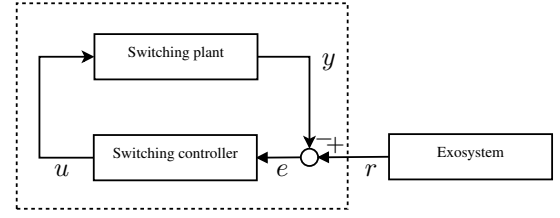


Fig. 1. Diagram of the closed-loop system.

appears to be mainly motivated by the goal of establishing connections between input selection and mode identifiability, in close analogy with the problem of input selection for systems parameter identification [11]. For configurations such as the one of Fig. 1, the question of mode identifiability has to be approached in a different way, of primary importance begin the issues of closed-loop stability and performance. Due to the limitations imposed by the control goals on input selection, two natural questions arise in connection with the control arrangement of Fig. 1:

- i) How the *control objectives* influence the possibility of identifying the active mode of the plant.
- ii) How the *control design* influence the possibility of identifying the active mode of the plant.

To render the exposition as simple as possible, we introduce from the very beginning the following assumptions.

*Assumption 1:* For any plant mode  $i \in \mathcal{N}$  and any controller mode  $j \in \mathcal{M}$ , the corresponding time-invariant systems  $\mathcal{P}_i$  and  $\mathcal{C}_j$  are controllable and observable.  $\square$

As for the control objectives, next assumptions naturally follow from internal-model-based control [14]. Consider a left and a right polynomial matrix fraction description (PMFD) of the  $i$ -th plant,

$$\mathcal{P}_i(s) := U_i^{-1}(s) Q_i(s) = \mathcal{Q}_i(s) \mathcal{U}_i^{-1}(s), \quad (4)$$

where, for each index  $i \in \mathcal{N}$ ,  $U_i$  and  $Q_i$  [ $\mathcal{U}_i$  and  $\mathcal{Q}_i$ ] are left [right] coprime polynomial matrices of appropriate dimensions. Likewise, let

$$\mathcal{C}_j(s) := R_j^{-1}(s) S_j(s) = \mathcal{S}_j(s) \mathcal{R}_j^{-1}(s). \quad (5)$$

be left and right PMFDs of the  $j$ -th plant controller, where, for each  $j \in \mathcal{M}$ ,  $R_j$  and  $S_j$  [ $\mathcal{R}_j$  and  $\mathcal{S}_j$ ] are left [right] coprime polynomial matrices of appropriate dimensions.

*Assumption 2:* All the eigenvalues of the transition matrix  $E$  have zero real part and multiplicity one in the minimal polynomial ( $r$  is a combination of sinusoidal signals).  $\square$

*Assumption 3:*  $Q_i(\lambda)$  is full-rank for every  $i \in \mathcal{N}$  and every  $\lambda \in \text{spec}\{E\}$ , where  $\text{spec}\{\cdot\}$  denotes spectrum (none of the eigenvalues of  $E$  is a *transmission zeros* of  $\mathcal{P}_i$ ).  $\square$

*Assumption 4:*  $R_j(\lambda) = 0$  for every  $j \in \mathcal{M}$  and every  $\lambda \in \text{spec}\{E\}$  (each controller embeds an *internal model* of the exosystem).  $\square$

### III. TRADE-OFFS BETWEEN MODE-OBSERVABILITY AND CONTROL OBJECTIVES

In this section, we enter into the detail of questions i) and ii) raised above. In Section III-A we introduce a notion of closed-loop mode-observability; Section III-B discusses fundamental limitations imposed by the control objectives on closed-loop mode-observability; finally, Section III-C discusses closed-loop mode-observability properties achievable by means of control design.

#### A. Closed-loop Mode-observability

Let  $w := (x^\top q^\top)^\top$  denote the state of the closed-loop switching system, and let  $z := (u^\top e^\top)^\top$  denote the corresponding output. We obtain

$$\begin{cases} \dot{w} = A_{\rho/\sigma}^{cl} w + B_{\rho/\sigma}^{cl} r \\ z = C_{\rho/\sigma}^{cl} w + D_{\sigma}^{cl} r \end{cases}, \quad w(t_0) = w_0, \quad (6)$$

where

$$\begin{aligned} A_{i/j}^{cl} &:= \begin{pmatrix} A_i - B_i K_j C_i & B_i H_j \\ -G_j C_i & F_j \end{pmatrix}, \quad B_{i/j}^{cl} := \begin{pmatrix} B_i K_j \\ G_j \end{pmatrix} \\ C_{i/j}^{cl} &:= \begin{pmatrix} -K_j C_i & H_j \\ -C_i & 0 \end{pmatrix}, \quad D_j^{cl} := \begin{pmatrix} K_j \\ I \end{pmatrix}. \end{aligned}$$

Let  $w_{i/j}(t, t_0, w_0, p_0)$  and  $z_{i/j}(t, t_0, w_0, p_0)$  denote the state and, respectively, the output response of (6) at  $t$  when the controller switching signal is  $\sigma(\tau) = j$  for all  $\tau \in [t_0, t]$ , the plant switching signal is  $\rho(\tau) = i$  for all  $\tau \in [t_0, t]$ , the closed-loop system initial state is  $w(t_0) = w_0$ , and the initial state of the exosystem is  $p(t_0) = p_0$ .

*Definition 1:* For system (6), two different plant modes  $i, \ell \in \mathcal{N}$  are said to be *closed-loop distinguishable* if

$$z_{i/j}(\cdot, t_0, w_0, p_0) \neq z_{\ell/j}(\cdot, t_0, w'_0, p_0) \quad \text{a.e. on } \mathcal{I},$$

for any  $\mathcal{I} := [t_0, t_0 + T)$ ,  $T > 0$ , any  $j \in \mathcal{M}$ , and any nonzero vector  $(w_0^\top w'_0{}^\top p_0^\top)^\top$ . System (6) is said to be *closed-loop mode-observable* if any two different plant modes are closed-loop distinguishable.  $\square$

By mode-observability we therefore refer to the possibility of discerning which plant modes  $i \in \mathcal{N}$  could have produced the measured data  $z$ , collected with the controller  $C_j$  in the feedback loop.

In order to establish a link between mode-observability and control objectives, we rewrite  $z_{i/j}(t, t_0, w_0, p_0)$  in a form better suited for analysis purposes.

*Definition 2:* The feedback loop  $(\mathcal{P}_i/\mathcal{C}_j)$  admits a steady-state response if there exists a matrix  $W_{i/j}$  such that, for every  $p_0$ , there exists a  $w_0^*$  such that

$$w_{i/j}(t, t_0, w_0^*, p_0) = W_{i/j} e^{E(t-t_0)} p_0, \quad (7)$$

for any  $t \geq t_0$ . Furthermore, provided that  $W_{i/j}$  exists, the feedback loop  $(\mathcal{P}_i/\mathcal{C}_j)$  is said to be in steady-state on  $\mathcal{I}$  if  $w_0 = W_{i/j} p_0$ .  $\square$

It is not difficult to verify that the existence of a steady-state response for  $(\mathcal{P}_i/\mathcal{C}_j)$  is equivalent to the existence of a matrix  $W_{i/j}$  that solves the Sylvester equation [15]

$$A_{i/j}^{cl} W_{i/j} + B_{i/j}^{cl} L = W_{i/j} E. \quad (8)$$

Let  $\varphi_{i/j}$  denote the characteristic polynomial of the closed-loop  $(\mathcal{P}_i/\mathcal{C}_j)$ . In view of Assumption 3 and 4, it is immediate to see that

$$\begin{aligned} \varphi_{i/j}(\lambda) &= \det \begin{pmatrix} Q_i(s) & U_i(s) \\ -R_j(s) & S_j(s) \end{pmatrix} \Big|_{s=\lambda} \\ &= \det(Q_i(\lambda)) \det(S_j(\lambda)) \neq 0 \end{aligned}$$

for every  $i \in \mathcal{N}$ ,  $j \in \mathcal{M}$ , and every  $\lambda \in \text{spec}\{E\}$ . We can therefore conclude that

$$\text{spec}\{A_{i/j}^{cl}\} \cap \text{spec}\{E\} = \emptyset \quad (9)$$

for every  $i \in \mathcal{N}$ ,  $j \in \mathcal{M}$ .

The main implication of (9) is that for each feedback loop  $(\mathcal{P}_i/\mathcal{C}_j)$  the following properties hold: there always exists a steady-state solution; and the steady-state solution is unique. These properties are indeed a direct consequence of the fact that (9) implies existence and uniqueness of the solution  $W_{i/j}$  in (8) (e.g. see [16]). This leads to the desired alternative expression for  $z_{i/j}(t, t_0, w_0, p_0)$ . Indeed, exploiting (8), simple calculations yield

$$\begin{aligned} z_{i/j}(t, t_0, w_0, p_0) &= \\ C_{i/j}^{cl} e^{A_{i/j}^{cl}(t-t_0)} (w_0 - W_{i/j} p_0) + Z_{i/j} e^{E(t-t_0)} p_0, \end{aligned} \quad (10)$$

where  $Z_{i/j} := C_{i/j}^{cl} W_{i/j} + D_j^{cl} L$ .

The advantage of using (10) is that we can decompose the output in terms of a “transient” response plus a “steady-state” response, the latter being obtained with  $w_0 = W_{i/j} p_0$ . As shown below, this makes it possible to obtain closed-loop mode-observability conditions that are more directly related to control issues. Let

$$G_{i,\ell/j}(\mathcal{I}) := \int_{\mathcal{I}} \begin{pmatrix} \mathcal{H}_{i,\ell/j}^\top \mathcal{H}_{i,\ell/j} & \mathcal{H}_{i,\ell/j}^\top \mathcal{L}_{i,\ell/j} \\ \mathcal{L}_{i,\ell/j}^\top \mathcal{H}_{i,\ell/j} & \mathcal{L}_{i,\ell/j}^\top \mathcal{L}_{i,\ell/j} \end{pmatrix} (\tau) d\tau$$

where

$$\mathcal{H}_{i,\ell/j}(\tau) := \begin{pmatrix} \Psi_{i/j}(\tau, t_0) & -\Psi_{\ell/j}(\tau, t_0) \end{pmatrix}$$

$$\Psi_{i/j}(\tau, t_0) := C_{i/j}^{cl} e^{A_{i/j}^{cl}(\tau-t_0)}$$

and

$$\mathcal{L}_{i,\ell/j}(\tau) := (Z_{i/j} - Z_{\ell/j}) e^{E(\tau-t_0)}$$

Exploiting (10), simple manipulations show that Def. 2 can be rephrased by saying that two different plant modes  $i$  and  $\ell$  are closed-loop distinguishable if and only if

$$\begin{pmatrix} \tilde{w}_0^\top & \tilde{w}_0'^\top & p_0^\top \end{pmatrix}^\top G_{i,\ell/j}(\mathcal{I}) \begin{pmatrix} \tilde{w}_0 \\ \tilde{w}_0' \\ p_0 \end{pmatrix} \neq 0 \quad (11)$$

for any  $\mathcal{I}$ ,  $j \in \mathcal{M}$ , and any nonzero vector  $(\tilde{w}_0^\top \tilde{w}_0'^\top p_0^\top)^\top$  where

$$\tilde{w}_0 := w_0 - W_{i/j} p_0, \quad \tilde{w}_0' := w_0' - W_{\ell/j} p_0.$$

Next proposition can be stated.

*Proposition 1:* Let Assumptions 1-4 hold, and consider any nonzero output data sequence  $z$  produced by the feedback loop (6) with plant mode  $\rho(\tau) = i$  for all  $\tau \in \mathcal{I}$  and controller mode  $\sigma(\tau) = j$  for all  $\tau \in \mathcal{I}$ . Then the plant mode can be uniquely identified on  $\mathcal{I}$  from the output data sequence  $z$  if and only if  $G_{i,\ell/j}(\mathcal{I})$  is positive definite for any candidate plant mode  $\ell \in \mathcal{N}$  different from  $i$ .  $\square$

*Remark 1: (Connections with Least-Square Observability)*— While Definition 2 and the subsequent arguments are independent of the method used to recover the mode, they implicitly define a simple approach in this regard. Indeed, in order to identify which plant mode could have produced the data  $z$  collected with the controller  $\mathcal{C}_j$ , one can solve, for each candidate plant mode  $i$ , the following minimization problem

$$\begin{aligned} \delta_{i/j}(z(\cdot), \mathcal{I}) &:= \min_w \|z(\cdot) - z_{i/j}(\cdot, t_0, w, p_0)\|_{2,\mathcal{I}} \\ &= \min_w \|\zeta_{i/j}(\cdot, t_0) - \Psi_{i/j}(\cdot, t_0) w\|_{2,\mathcal{I}} \end{aligned} \quad (12)$$

where

$$\begin{aligned} \zeta_{i/j}(t, t_0) &:= z(t) - C_{i/j}^{cl} \int_{t_0}^t e^{A_{i/j}^{cl}(t-\tau)} B_{i/j}^{cl} r(\tau) d\tau \\ &\quad - D_j^{cl} r(t) \end{aligned}$$

which is a standard least-square minimization problem.  $\square$

The main advantage of resorting to a closed-loop mode-observability condition based on  $G_{i,\ell/j}(\mathcal{I})$  is that such a formulation makes it possible to approach the question of mode-observability by looking at the contributions of transient and steady-state responses separately. As we will see, this can be used to establish a number of existing trade-offs between closed-loop mode-observability and control objectives. In particular, we will show that, while it may be impossible to uniquely recover the plant mode when the closed-loop evolves along its steady-state trajectory, suitable design conditions do exist under which it is always possible to uniquely recover the plant mode when the closed-loop evolves along non steady-state trajectories.

### B. Limitations imposed by the Control Objectives on Closed-Loop Mode-Observability

We begin our discussion by recalling that for any feedback interconnection  $(\mathcal{P}_i/\mathcal{C}_j)$ , the steady-state output response can be obtained by letting  $w_0 = W_{i/j} p_0$ . Hence, a necessary prerequisite for two plant modes  $i$  and  $\ell$  to be distinguishable is that the steady-state responses generated by the closed-loops  $(\mathcal{P}_i/\mathcal{C}_j)$  and  $(\mathcal{P}_\ell/\mathcal{C}_j)$  be different a.e. on  $\mathcal{I}$  for any  $j \in \mathcal{M}$ .

Given a pair of matrices  $(M, N)$  with  $M \in \mathbb{R}^{h \times l}$  and  $N \in \mathbb{R}^{l \times l}$ , let

$$\mathcal{O}_{(M,N)} := \begin{pmatrix} M^\top & (MN)^\top & \dots & (M(N)^{l-1})^\top \end{pmatrix}^\top$$

denote its observability matrix. Exploiting (11), we have at once a necessary prerequisite for closed-loop mode-observability.

*Proposition 2:* Under the same assumptions and conditions of Proposition 1, the plant mode can be uniquely identified on  $\mathcal{I}$  only if  $\mathcal{O}_{(Z_{i/j}-Z_{\ell/j}, E)}$  is full-rank for any candidate plant modes  $\ell \in \mathcal{N}$  different from  $i$ .  $\square$

Although the requirement expressed in Proposition 2 seemingly depends on both plant and controller modes, actually, mode-observability along steady-state trajectories does only depend on the *open-loop* plant features. This can be seen by resorting to modal analysis. To this end, consider that by virtue of Assumption 2 there exists a similarity transformation  $\mathcal{T} \in \mathbb{C}^{n_p \times n_p}$  of the form  $\mathcal{T} := (v_1 v_2 \dots v_{n_p})$  such that  $\mathcal{T}^{-1} E \mathcal{T} = \Lambda_E$  with  $\Lambda_E := \text{diag}\{\lambda_1, \dots, \lambda_{n_p}\}$ . Then, by letting  $\xi_0 := \mathcal{T}^{-1} p_0 = (\xi_{10}, \dots, \xi_{n_p 0})^\top$ , the reference signal can be written as

$$r(t) = L \sum_{k=1}^{n_p} \xi_{k0} e^{\lambda_k(t-t_0)} v_k.$$

The steady-state output response of  $(\mathcal{P}_i/\mathcal{C}_j)$  can be therefore written as

$$z_{i/j}(t, t_0, W_{i/j} p_0, p_0) = \sum_{k=1}^{n_p} \xi_{k0} e^{\lambda_k(t-t_0)} \begin{pmatrix} u_{i/j}^k \\ c_{i/j}^k \end{pmatrix},$$

where  $u_{i/j}^k$  and  $c_{i/j}^k$  satisfy

$$\begin{pmatrix} Q_i(\lambda_k) & U_i(\lambda_k) \\ -R_j(\lambda_k) & S_j(\lambda_k) \end{pmatrix} \begin{pmatrix} u_{i/j}^k \\ c_{i/j}^k \end{pmatrix} = \begin{pmatrix} U_i(\lambda_k) \\ 0 \end{pmatrix} L v_k.$$

from which one can easily verify that

$$c_{i/j}^k = 0, \quad u_{i/j}^k = u_i^k = Q_i(\lambda_k)^{-1} U_i(\lambda_k) L v_k. \quad (13)$$

Notice that, even if Assumption 4 is controller-dependent, it may become a necessary prerequisite for the control arrangement when asymptotic tracking cannot be taken care by the internal plant modes. In view of this fact, one concludes that two plant modes  $i$  and  $\ell$  are indistinguishable in steady-state if and only if there exists at least one index  $k \in \{1, \dots, n_p\}$  for which

$$Q_i(\lambda_k)^{-1} U_i(\lambda_k) L v_k = Q_\ell(\lambda_k)^{-1} U_\ell(\lambda_k) L v_k.$$

Under such circumstances, *no matter how* the controller  $\mathcal{C}_j$  has been chosen, it is impossible to determine whether a steady-state output response is been generated by  $\mathcal{P}_i$  or  $\mathcal{P}_\ell$ .

### C. Closed-Loop Mode-Observability Properties achievable by means of Control Design

While previous analysis indicates that it may be impossible to uniquely recover the plant mode when the closed-loop is in steady-state, it is simple to see that suitable *control design* conditions do exist that ensure plant mode identification along all non steady-state trajectories.

In close analogy with Proposition 2, one first observes that a necessary prerequisite for two plant modes  $i$  and  $\ell$  to be distinguishable is that the transient responses generated by  $(\mathcal{P}_i/\mathcal{C}_j)$  and  $(\mathcal{P}_\ell/\mathcal{C}_j)$  be different a.e. on  $\mathcal{I}$  for any  $j \in \mathcal{M}$ . More precisely, distinguishability along non steady-state trajectories amounts to requiring that

$$C_{i/j}^{cl} e^{A_{i/j}^{cl}(\tau-t_0)} \tilde{w}_0 \neq C_{\ell/j}^{cl} e^{A_{\ell/j}^{cl}(\tau-t_0)} \tilde{w}'_0, \quad \text{a.e. on } \mathcal{I},$$

for any non-zero vector  $(w_0^\top w'_0{}^\top)^\top$ , where we recall that  $\tilde{w}_0 = w_0 - W_{i/j} p_0$  and  $\tilde{w}'_0 = w'_0 - W_{\ell/j} p_0$ . Note that the vectors  $\tilde{w}_0$  and  $\tilde{w}'_0$  are arbitrary since both  $w_0$  and  $w'_0$  are such. By letting

$$\Theta_{i,\ell/j} := \left( \mathcal{O}_{(C_{i/j}^{cl}, A_{i/j}^{cl})} - \mathcal{O}_{(C_{\ell/j}^{cl}, A_{\ell/j}^{cl})} \right) \quad (14)$$

we have at once the following result.

**Proposition 3:** Under the same assumptions and conditions of Proposition 1, the plant mode can be uniquely identified on  $\mathcal{I}$  only if  $\Theta_{i,\ell/j}$  is full-rank for any candidate plant modes  $\ell \in \mathcal{N}$  different from  $i$ .  $\square$

Consider now that  $\Theta_{i,\ell/j}$  coincides with the observability matrix of the autonomous system

$$\begin{aligned} \dot{\chi}(t) &= \begin{pmatrix} A_{i/j}^{cl} & 0 \\ 0 & A_{\ell/j}^{cl} \end{pmatrix} \chi(t) \\ v(t) &= \begin{pmatrix} C_{i/j}^{cl} & -C_{\ell/j}^{cl} \end{pmatrix} \chi(t) \end{aligned}$$

obtained from the parallel connection of  $(\mathcal{P}_i/\mathcal{C}_j)$  and  $(\mathcal{P}_\ell/\mathcal{C}_j)$ . One can therefore avail of the following results which descends directly from the observability properties of composite systems [17].

**Proposition 4:** The matrix  $\Theta_{i,\ell/j}$  is full-rank if the closed-loop characteristic polynomials  $\varphi_{i/j}(s)$  and  $\varphi_{\ell/j}(s)$  are coprime. Moreover, for single-input single-output systems, this is also necessary.  $\square$

Proposition 4 indicates that it is possible to ensure distinguishability in non steady-state provided that for any two different plant modes  $i, \ell \in \mathcal{N}$  and any controller mode  $j \in \mathcal{M}$  the closed-loop polynomials  $\varphi_{i/j}$  and  $\varphi_{\ell/j}$  have no common roots. Proposition 4 therefore suggests how the controllers have to be designed in order for plant mode identification to be possible when the system evolves along non steady-state trajectories. This aspect has some interesting implications. In fact, by recalling that the spectra of  $A_{i/j}^{cl}$  and  $E$  are disjoint for all  $i \in \mathcal{N}$  and  $j \in \mathcal{M}$ , we have that the transient response  $C_{i/j}^{cl} e^{A_{i/j}^{cl}(t-t_0)} (w_0 - W_{i/j} p_0)$

and the steady-state response  $Z_{i/j} e^{E(t-t_0)} p_0$  are linearly independent on every interval  $\mathcal{I}$ . Thus, if we let <sup>1</sup>

$$\mathcal{S}_{i/j}(\mathcal{I}) := \left\{ \hat{z} \in \mathcal{L}_2(\mathcal{I}) : \hat{z}(\cdot) = z_{i/j}(\cdot, t_0, \hat{w}, p_0) \right. \\ \left. \text{on } \mathcal{I}, \text{ for some } \hat{w} \in \mathbb{R}^{n_x+n_q} \right\}. \quad (15)$$

denote the set of all possible measured data on  $\mathcal{I}$  associated with a plant mode  $i$  and a controller mode  $j$ , whenever the controller family is designed so as to satisfy the conditions of Proposition 4, the set of possible trajectories common to any two different plant modes reduces to

$$\mathcal{S}_{i/j}(\mathcal{I}) \cap \mathcal{S}_{\ell/j}(\mathcal{I}) = \begin{cases} Z_{i/j} e^{E(t-t_0)} p_0, & \text{if } p_0 \in \ker \left\{ \mathcal{O}_{(Z_{i/j}-Z_{\ell/j}, E)} \right\} \\ \emptyset, & \text{otherwise} \end{cases} \quad (16)$$

### IV. MODE-OBSERVABILITY UNDER MEASUREMENT NOISE

In this section, we discuss some questions concerned with closed-loop mode-observability in the presence of persistent disturbances. To this end, suppose that the measurement equations in (1) are affected by an additive noise  $v$ . Accordingly, we can rewrite the feedback loop dynamics of (6) as

$$\begin{cases} \dot{w} = A_{\rho/\sigma}^{cl} w + B_{\rho/\sigma}^{cl} (r - v) \\ z = C_{\rho/\sigma}^{cl} w + D_{\sigma}^{cl} (r - v) \end{cases}. \quad (17)$$

The main complication arising in this case concerns the fact that even when both the plant mode as well as the controller mode take on a constant value over  $\mathcal{I}$ , the presence of the noise  $v$  prevents one from applying previous results since the output data sequence  $z$  need not longer belong to (15). Nonetheless, even in this case, it is possible to establish a number of interesting connections between closed-loop mode-observability and control.

Consider any nonzero output data sequence  $z$  produced by the feedback loop (17) with plant mode  $\rho(\tau) = i$  for all  $\tau \in \mathcal{I}$  and controller mode  $\sigma(\tau) = j$  for all  $\tau \in \mathcal{I}$ . Decompose  $z$  as follows

$$z(\cdot) = z^{(n)}(\cdot) + z^{(fr)}(\cdot) + z^{(fv)}(\cdot)$$

where  $z^{(n)}(t)$  is the natural response while  $z^{(fr)}(t)$  and  $z^{(fv)}(t)$  denote the forced response due to  $r$  and  $v$ , respectively. As pointed out in the previous sections, when mode observability holds the plant mode can be reconstructed by observing the natural response and the forced response due to  $r$ . By exploiting triangular inequality and recalling that  $\delta_{i/j}(z^{(n)}(\cdot) + z^{(fr)}(\cdot), \mathcal{I}) = 0$ , it is immediate to see that, when the estimation scheme suggested in Remark 1 is

<sup>1</sup>  $\mathcal{L}_2(\mathcal{I})$  denotes the sets of square integrable time functions on  $\mathcal{I}$ .

adopted, a sufficient condition for identifying the mode in the presence of noises is that

$$\delta_{\ell/j}(z^{(n)}(\cdot) + z^{(fr)}(\cdot), \mathcal{I}) > 2 \|z^{(fv)}(\cdot)\|_{2,\mathcal{I}} \quad (18)$$

for any candidate plant mode  $\ell \in \mathcal{N}$  different from  $i$ .

Let now

$$\kappa_B := \max_{i \in \mathcal{N}, j \in \mathcal{M}} \|B_{i/j}^{cl}\|, \\ \kappa_C := \max_{i \in \mathcal{N}, j \in \mathcal{M}} \|C_{i/j}^{cl}\|, \quad \kappa_D := \max_{i \in \mathcal{N}, j \in \mathcal{M}} \|D_{i/j}^{cl}\|,$$

and observe that, since the sets  $\mathcal{N}$  and  $\mathcal{M}$  are finite, there exist positive reals  $\theta$  and  $\eta$  such that  $\|e^{A_{i/j}^{cl}t}\| \leq \theta e^{\eta t}$  for all  $t \in \mathbb{R}_+$ ,  $i \in \mathcal{N}$  and  $j \in \mathcal{M}$ . Accordingly,

$$|z^{(fv)}(t)| \leq \kappa_C \kappa_B \theta \int_{\mathcal{I}} e^{\rho(t-\tau)} |v(\tau)| d\tau + \kappa_D |v(t)|$$

for all  $t \in \mathcal{I}$ . Basically, this means that, over any interval  $\mathcal{I}$ , it is possible to bound the contribution of the forced response due to  $v$  by

$$\|z^{(fv)}(\cdot)\|_{2,\mathcal{I}} \leq \psi(\mathcal{I}) \|v(\cdot)\|_{\infty,\mathcal{I}}$$

where

$$\psi(\mathcal{I}) := \sqrt{T} \left( \kappa_B \kappa_C \theta \frac{e^{\eta T} - 1}{\eta} + \kappa_D \right).$$

By recalling (11) and Proposition 1, under closed-loop mode-observability, the smallest eigenvalue of  $G_{i,\ell/j}(\mathcal{I})$  is strictly positive for any any candidate plant mode  $\ell \in \mathcal{N}$  different from  $i$ . More precisely, let

$$\omega_{min}(\mathcal{I}) := \min_{i, \ell \in \mathcal{N}, i \neq \ell, j \in \mathcal{M}} \lambda_{min} \{G_{i,\ell/j}(\mathcal{I})\}$$

where  $\lambda_{min}$  denotes minimum eigenvalue. Thus, we have that a sufficient condition for (18) to hold is

$$\sqrt{|\tilde{w}_0|^2 + |p_0|^2} \geq \frac{2\psi(\mathcal{I})\|v(\cdot)\|_{\infty,\mathcal{I}}}{\sqrt{\omega_{min}(\mathcal{I})}}. \quad (19)$$

The following observation can be made about (19): the larger the closed-loop initial state  $w_0$  or the exosystem initial state  $p_0$  the easier the plant mode identification task. As for the former, this can be thought of as saying that mode detection is easier under closed-loop *divergence trends*. This fact is very not surprising since, close to what happens in supervisory adaptive control, it is usually easier to discriminate stabilizing from destabilizing controllers, rather than between stabilizing controllers solely. As for  $p_0$ , (19) indicates that, while the presence of the exosystem can prevent one from achieving plant mode identification under all circumstances, the presence of high *signal-to-noise ratios* may counteract the effect of the noise. In such a case, the reference itself becomes cooperative to the plant mode identification task.

## V. CONCLUSIONS

In this paper we described recent progress in the study of switching linear systems governed by unknown switching sequences. It was shown that for switching linear systems in feedback form, trade-offs between mode-observability and control objectives may arise. In particular, it was shown that the presence of control objectives such as asymptotic tracking can make the plant mode identification task impossible to solve, thus revealing the possible existence of conflicting objectives. It was also shown, nonetheless, that suitable control design conditions do exist under which it is possible to ensure mode-observability along all closed-loop trajectories except those corresponding to steady-state offset-free tracking.

## REFERENCES

- [1] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, 2003.
- [2] R. Vidal, A. Chiuso, S. Soatto, S. Sastry, Observability of linear hybrid systems, in: F. Wiedijk, O. Maler, A. Pnueli (Eds.), *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science 2623, Springer, 2003, pp. 523–539.
- [3] M. Babaali, G. J. Pappas, Observability of switched linear systems in continuous time, in: M. Morari, L. Thiele (Eds.), *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science 3414, Springer-Verlag, Berlin Heidelberg, 2005, pp. 103–117.
- [4] M. Babaali, M. Egerstedt, Observability of switched linear systems, in: R. Alur, G. J. Pappas (Eds.), *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science, Springer, 2004, pp. 48–63.
- [5] M. Baglietto, G. Battistelli, L. Scardovi, Active mode observability of switching linear systems, *Automatica* 43 (8) (2007) 1442–1449.
- [6] M. Baglietto, G. Battistelli, L. Scardovi, Active mode observation of switching systems based on set-valued estimation of the continuous state, *International Journal of Robust and Nonlinear Control* 19 (14) (2009) 1521–1540.
- [7] F. Blanchini, S. Miani, F. Mesquine, A separation principle for linear switching systems and parametrization of all stabilizing controllers, *IEEE Transactions on Automatic Control* 54 (2) (2009) 279–292.
- [8] J.-W. Lee, G. E. Dullerud, Uniform stabilization of discrete-time switched and markovian jump linear systems, *Automatica* 42 (2) (2006) 205–218.
- [9] D. Xie, Y. Wu, Stabilisability of switched linear systems with time-varying delay in the detection of switching signals, *IET Control Theory Applications* 3 (4) (2009) 404–410.
- [10] M. Blanke, M. K. J. Lunze, M. Staroswiecki, *Diagnosis and Fault-Tolerant Control*, Springer-Verlag, 2nd Ed., 2006.
- [11] M. Grewal, K. Glover, Identifiability of linear and nonlinear systems, *IEEE Transactions on Automatic Control* 21 (6) (1976) 833–837.
- [12] G. Battistelli, On adaptive stabilization of mode-observable switching linear systems, in: *Proc. of the 18th IFAC World Congress*, Milan, Italy, 2011.
- [13] M. Baglietto, G. Battistelli, P. Tesi, Set-point tracking for mode-observable switching linear systems, in: *Proc. of the 50th IEEE Conference on Decision and Control*, Orlando, Florida, USA, 2011.
- [14] A. Isidori, L. Marconi, A. Serrani, *Robust Autonomous Guidance. An Internal Model Approach* (Advances in Industrial Control), Springer Verlag, 2003.
- [15] G. Basile, G. Marro, *Controlled and conditioned invariants in linear system theory*, Prentice Hall, 1992.
- [16] A. Ostrowski, H. Schneider, Some theorems on the inertia of general matrices, *J. of Math. Analysis and Applications* 4 (1962) 72–84.
- [17] C. Chen, C. Desoer, Controllability and observability of composite systems, *IEEE Trans. on Automatic Control* 12 (4) (1967) 402–409.